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A Toom Rule That Increases the Thickness of Sets

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Toom's north-east-self voting cellular automaton rule R is known to suppress small minorities. A variant, R^+, is also known to turn an arbitrary initial configuration into a homogeneous one (without changing the ones that were homogeneous to start with). Here it is shown that R^+ always increases a certain property of sets called thickness. This result is intended as a step toward a proof of the fast convergence toward consensus under R^+. The latter is observable experimentally, even in the presence of some noise.

KEY WORDS: Cellular automata; Toom's rule; statistical mechanics.

1. INTRODUCTION

1.1. Cellular Automata

Cellular automata are useful as models of some physical and biological phenomena and of computing devices. To define a cellular automaton, first a set $\mathbf{S}$ of possible *local states* is given. In the present paper, this is the two-element set $\{0, 1\}$. Then, a set $\mathbf{W}$ of *sites* is given. In the present paper, this is the two-dimensional integer lattice $\mathbf{Z}^2$. A *configuration*, or *global state*, x over a subset B of $\mathbf{W}$ is a function that assigns a state $x[p] \in \mathbf{S}$ to each element p of B. An *evolution* $x[t, p]$ over a time interval $t_1,..., t_2$ and a set B of sites is a function that assigns a global state $x[t, \cdot]$ over B to all $t = t_1,..., t_2$. A *neighborhood* is a finite set $G = \{g_1,..., g_k\}$ of elements of $\mathbf{Z}^2$. A *transition rule* is a function M: $\mathbf{S}^k \to \mathbf{S}$. An evolution $x[t, p]$ is called a *trajectory* of the transition rule M if the relation

$$x[t+1, p] = M(x[t, p+g_1],..., x[t, p+g_k]) \tag{1.1}$$

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holds for all t, p. To obtain a trajectory over the whole space **W**, we can start from an arbitrary initial configuration $x[0, \cdot]$ and apply the local transformation (1.1) to get the configurations $x[1, \cdot]$, $x[2, \cdot]$,.... The rule (1.1) is analogous to a partial differential equation.

Most work done with cellular automata is experimental. It seems to follow from the nature of the broader subject ("chaos") involving the iteration of transformations that exact results are difficult to obtain. The reason seems to be that a trajectory of an arbitrary transition rule is like an arbitrary computation, and most nontrivial problems concerning arbitrary computations are undecidable.

Most of the exact work concerns *probabilistic* cellular automata, i.e., ones in which the value of the transition rule M is a probability distribution over **S**. As a simple example, let us consider a deterministic rule M and an initial configuration $x[0, \cdot]$. We begin to apply the relation (1.1) to compute $x[t+1, p]$, but occasionally (these occasions occur, say, independently with a low probability ρ), we will violate the rule and take a different value for $x[t+1, p]$. The random process obtained in this way can be called, informally, a *ρ-perturbation* of the trajectory obtained from $x[0, \cdot]$.

The most thoroughly investigated problem concerning probabilistic cellular automata is a problem analogous to the *phase transition* problem of equilibrium systems (like the Ising model of ferromagnetism). Given a probabilistic transition rule, the problem corresponding to the phase-transition problem of equilibrium systems is whether the evolution erases all information concerning the initial configuration. In the case, it is said that the system does not have a phase transition.

The known equilibrium models that exhibit phase transition are not known to be stable: if the parameters are slightly perturbed (e.g., an outside magnetic field turned on), the phase transition might disappear. In contrast, there are cellular automata exhibiting a stable phase transition. If was not a trivial problem to find such cellular automata. Indeed, let us look at probabilistic rules obtained by the perturbation of a deterministic one. If the rule is the identity, i.e., $x[t+1, p] = x[t, p]$, then this rule remembers the initial configuration, as long as it is not perturbed. If it is perturbed appropriately, then the information in the configuration $x[t, \cdot]$ about $x[0, \cdot]$ converges fast to 0. Also, most local majority voting rules seem to lose all information fast when perturbed appropriately.

1.2. Toom's Rule

The first rules exhibiting stable phase transition were found by Andrei Toom. A general theory of them is given in ref. 3.

One of Toom's rules is defined with the neighborhood

$$G = \{(0, 0), (0, 1), (1, 0)\}$$

and the transition function M which is the majority function $\mathrm{Maj}(x, y, z)$. In other words, an evolution $x[t, p]$ is a trajectory of Toom's rule R if for all t, p where it is applicable, the following relation holds:

$$x[t+1, p] = \mathrm{Maj}(x[t, p], x[t, p+(0, 1)], x[t, p+(1, 0)]$$

We will also write

$$x[t+1, \cdot] = R(x[t, \cdot\,])$$

The rule R says that to compute the next value in time of trajectory x at some site, we have to compute the majority of the current value at the site and its northern and eastern neighbors.

For $s = 0, 1$, let h_s be the homogeneous configuration for which $h_s[p] = s$ for all sites p. The north-east-self voting rule R is known to suppress small minorities, even in the presence of noise. If started from a homogeneous configuration, then the one-bit information saying whether this configuration was h_0 or h_1 is preserved.

There are many variants of the rule R, all of which have the noise-suppressing property. One of these was used in ref. 2 to define a simple three-dimensional rule that not only can store an infinite amount of information about the initial state, but can also simulate the trajectory of an arbitrary one-dimensional deterministic rule, despite perturbation.

Given the simplicity of the rule R and its two stable configurations, it is natural to investigate the effect of repeated applications of R to an arbitrary configuration that is close to neither h_0 or h_1. We will identify a configuration x with the set of sites a, where $x[a] = 1$. Therefore we can talk about the application of R to a set.

Let $\mathbf{G}$ be the graph over $\mathbf{W}$ in which each point is conneceted to north, south, east, west, northwest, southeast. (The graph is undirected in the sense that with each directed edge, it also contains the reverse edge.) A subset of $\mathbf{W}$ is called *connected* if it is connected in G. Let $S = \bigcup_i S_i$ be a set with connected components S_i. The simple Lemma 3.1 proved later in this paper says that the rule R does not break up an does not connect the components S_i. For the plane $\mathbf{W} = \mathbf{Z}^2$, the simple Lemma 2.1 stated later says that Toom's rule "shrinks" each of the components, in terms of the size measure called span.

If the space $\mathbf{W}$ is the torus $\mathbf{Z}_n^2$, then the rule R still shrinks those connected sets that are isomorphic to subsets of $\mathbf{Z}^2$. These components will be called *simple*. Let us characterize them. The *increment* of each directed edge

$((a_1, b_1), (a_2, b_2))$ of G is the vector $(a_2 - a_1, b_2 - b_1)$. The absolute value of both coordinates of this vectors is $\leqslant 1$. The *total increment* along a path is the sum of the increments, *without reduction* mod n. A closed path (cycle) is *simple* if its total increment is 0. It is easy to see that a connected subset of $\mathbf{W}$ is simple if and only if it does not contain a nonsimple cycle. Now it is easy to verify the following theorem, proved in ref. 1.

Theorem 1.1. Let S be a subset of $\mathbf{W}$. The set $R^t(S)$ becomes eventually empty as $t \to \infty$ if and only if all components of S are simple.

Thus, the minimal sets not erased by the iteration of R are cycles that wind at least once around the torus. Toom's rule will not break up such cycles. It actually leaves many of them invariant, possibly shifting them.

1.3. Global Simplification

There is some interest in trying to find a variant to Toom's rule that still preserves the stability of the homogeneous states h_s but whose iterations force every configuration x eventually into some $\mathbf{H}(x) = h_0$ or h_1. Since there are only two homogeneous configurations, there will be configurations x, x' differing only in one site, where $\mathbf{H}(x) = h_0$ and $\mathbf{H}(x') = h_1$.

The main interest of such rules comes from the insight they give into the mechanism of *global simplification* of an arbitrary configuration necessary for such a property. Also of interest is the opportunity to investigate the noise sensitivity of the simplification, i.e., the size of the attraction domains.

A possible application of such a rule is in situations where a consensus must be forced from an arbitrary configuration. Gács[1] shows such a situation. Consensus problems, or, in a more extravagant terminology, Byzantine Generals' problems, are central in the area of computer science called distributed computing.

Consensus in the Absence of Failures. Theorem 1.1 above suggests a modification of the rule R with the desired property. Since the only configurations not erased by R are those containing nonsimple cycles, we should try to force all those cycles to h_1. This is achieved by biasing the rule R slightly in the direction of 1's, while still preserving the shrinking property given in Lemma 2.1. We obtain such a rule R^+ as follows. To compute the state $R^+(x)[p]$ of cell p after applying R^+ to the configuration x, apply the rule R twice to x, then take the maximum of the states of the neighbors $p, p + (0, -1), p + (-1, 0)$. The theorem below shows that R^+ indeed has the desired limiting consensus property. Of course, such a property is interesting only in connection with the presence of at least two stable configurations.

Theorem 1.2. There is a constant c such that the following holds. Let S be an arbitrary subset of $\mathbf{W}=\mathbf{Z}_n^2$. Then $(R^+)^{cn}(S)=h_0$ or h_1.

A proof was given in ref. 1. Let us sketch here a more direct proof. It uses the following lemma from ref. 1 saying that the rule R^+ first makes a set fat before erasing it. The proof is given, for the sake of completeness, in Section 2.2.

Lemma 1.1. Let S be a connected subset of $\mathbf{Z}^2$ with the property that $(R^+)^{2i}(S)\neq\varnothing$. Then $(R^+)^i(S)$ has at least $i^2/2$ elements.

The rule R^+ still has the property of rule R that it does not break up connected components. But, contrary to the rule R, it can join several components. The following lemma shows how the number of components gets smaller, provided no nonsimple component occurs. (If a nonsimple component occurs, then the rule R^+ blows it up anyway, in $\leqslant n$ steps, to occupy the whole space.)

Lemma 1.2. *Let* $C\subset\mathbf{W}=\mathbf{Z}_n^2$ *have* p *components, and* $D=(R^+)^{2i}(C)$ *have* q *components, all of them simple. If* $i\geqslant n(8/p)^{1/2}$, *then* $q\leqslant 0.75p$.

Proof. Let $C_1,\dots, C_p$ be the components of C and $D_1,\dots, D_q$ the components of D. Then there is a disjoint union $\{1,\dots, p\}=I_1\cup\cdots\cup I_q$ such that

$$D_j=(R^+)^{2i}\left(\bigcup_{k\in I_j} C_k\right)$$

Let K be the set of those j for which I_j consists of a single element i_j. These j belong to components C_{i_j} that are large enough and survive the $2i$ applications of R^+ without having to merge with other components. It follows from Lemma 1.1 that $|K|\,(i^2/2)\leqslant n^2$, i.e., $|K|\leqslant 2(n/i)^2$, since otherwise the number of elements of the set

$$(R^+)^i\left(\bigcup_{j\in K} C_{i_j}\right)$$

would be greater than the nuumber n^2 of elements of $\mathbf{W}$. Of course, we have $q-|K|\leqslant p/2$. Combining these, we have

$$q\leqslant p/2+2(n/i)^2$$

With $i\geqslant n(8/p)^{1/2}$, we have $q\leqslant 0.75p$. ∎

Proof of Theorem 1.2. Let us apply the last lemma repeatedly with

$$C^{k+1} = (R^+)^{2i_k}(C^k)$$

where p_k is the number of components of C_k, and $i_k = \lceil n(8/p_k)^{1/2} \rceil$. We get $p_{k+1} \leqslant 0.75 p_k$, hence the number of components decreases to 1 fast. The times $2i_k$ form approximately a geometric series in which even the largest term, obtained with $p_k = 2$, is at most $4n$. Therefore the sum of this series is still $\leqslant cn$ for an appropriate constant c. ∎

Consensus in the Presence of Failures. The sensitivity of the simplification property indicates difficulties if some violations of the rule are permitted, especially if these violations are not probabilistic but can be malicious. It still follows easily from Theorem 1.2 that R^+ achieves near consensus in $O(n^2)$ steps, even if $o(n)$ of the local transitions during this procedure were malicious failures. Indeed, in $\sim n^2$ steps, there is a time interval of size cn with the constant c of Theorem 1.2 without failures. During this interval, homogeneity is achieved, and given the stability of the rule R^+, the $o(n)$ failures cannot overturn it.

Eventually, we would like to show that near consensus is achieved under the same conditions already in $O(n)$ steps. This seems true, but difficult to prove. If failures are permitted, the monotonicity disappears. Components can not only be joined, but also split. The argument of Lemma 1.2 can be summarized thus:

> Small components either disappear or join to survive; therefore their number decreases fast. Large components become temporarily fat; therefore their number becomes small.

If components can also be split, then it is possible that small components join temporarily to survive, then failures split them again, and thus their number does not decrease.

Hope is given by an observation indicating a property that is a strengthening of Lemma 1.1. This lemma says that R^+ makes sets fatter before erasing them. The strengthening would say that the sets are made not only fat in the sense of containing many points, but also "thick," in the sense of becoming harder to split.

Informally, a set can be called k-thick if, for all $i < k$, cutting off a piece of size $6i$ from it, we need a cutting set of size approximately i. The present paper proves that R^+ indeed has a thickness-increasing property. Thus, if R^+ joins two large components and has k failure-free steps to work on the union, then the union cannot be split into two large components again by fewer than k failures. This application is the informal justification of the notion of thickness.

The proof of the thickness-increasing property is a lot of drudgery. Its claim to attention rests less on any aesthetic appeal than on being one of the few examples for the rigorous analysis of an interesting global behavior of an important cellular automaton.

2. SOME GEOMETRICAL DEFINITIONS

2.1. Tiles

Let us call a *tile* a triangle $Q(p)$ consisting of a point p and its northern and eastern neighbors (Fig. 1). Let us call p the *center* of the tile. We write

$$\begin{aligned} &e_1(p) = p, \qquad e_2(p) = p + (1, 0), \qquad e_3(p) = p + (0, 1) \\ &Q(p) = \{e_1(p), e_2(p), e_3(p)\} \end{aligned} \tag{2.1}$$

The "center" of the tile is thus really one of the corners. But it is better to view the center as identical with the tile itself. In illustrations, it is better to draw the tiles to be rotationally symmetric. The "center" of the tile is then the site at its bottom.

If the set S intersects a tile in at least two points, then we say that it *holds* the tile. The set $R(S)$ contains a point p iff S holds the tile with center p. We say that two tiles are *neighbors* if they intersect, or, equivalently, if their centers are neighbors. As mentioned above, it is convenient to think of the graph of tiles instead of the centers themselves, identifying the set $R(S)$ with the set of those tiles held by the elements of S. Let

$$Q(E) = \bigcup_{a \in E} Q(a)$$

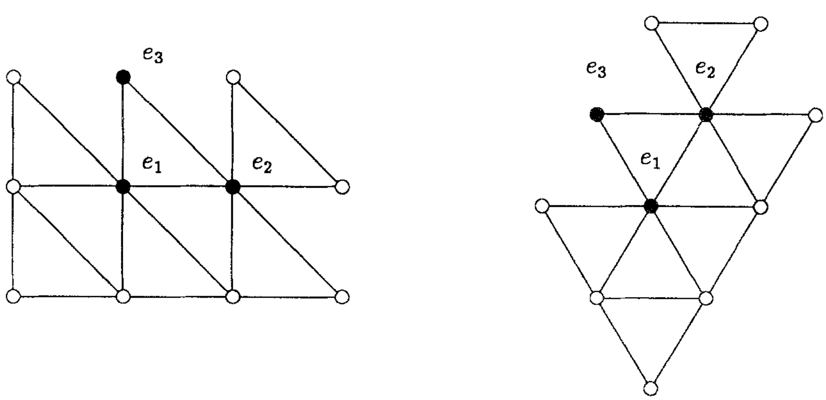

Fig. 1. The graph G and a tile, drawn in the original and in the symmetric fashion.

2.2. Triangles

Let us define the linear functions

$$L_1(\alpha, \beta) = -\alpha, \qquad L_2(\alpha, \beta) = -\beta, \qquad L_3(\alpha, \beta) = \alpha + \beta$$

The triangle $L(a, b, c)$ is defined as follows:

$$L(a, b, c) = \{p: L_1(p) \leqslant a, L_2(p) \leqslant b, L_3(p) \leqslant c\}$$

The deflation of the triangle $I = L(a, b, c)$ by the amount d is defined as follows:

$$D(I, d) = L(a-d, b-d, c-d)$$

The *span* of the above triangle is the length of its base, and is given by

$$\operatorname{span}(I) = a + b + c$$

For a set $\mathscr{I}$ of triangles we have

$$D(\mathscr{I}, d) = \{D(I, d): I \in \mathscr{I}\}$$

$$\operatorname{span}(\mathscr{I}) = \sum_{I \in \mathscr{I}} \operatorname{span}(I)$$

For a set E of lattice points, let $\operatorname{span}(E, d)$ be $\min \operatorname{span}(\mathscr{I})$, where the minimum is taken over sets $\mathscr{I}$ of triangles covering E with their d-deflation, i.e., for which $E \subset \bigcup D(\mathscr{I}, d)$. Here,

$$\bigcup D(\mathscr{I}, d) = \bigcup_{I \in \mathscr{I}} D(I, d)$$

Let

$$\mathrm{defl} = 2$$

We write

$$\operatorname{span}(E) = \operatorname{span}(E, 0)$$

$$\operatorname{Span}(E) = \operatorname{span}(E, \mathrm{defl})$$

The following lemma is easy to verify.

Lemma 2.1. For a connected set E of lattice points, let $\operatorname{span}(E, d) > 0$. Then

$$\operatorname{span}(R(E), d) = \operatorname{span}(E, d) - 1, \qquad \operatorname{span}(Q(E), d) = \operatorname{span}(E, d) + 1$$

The number span$(E, 1/3)$ will be called the *discrete span of E.* Then discrete span of a single point is 1. Two points are neighbors in $\mathbf{G}$ iff the discrete span of their pair is $\leqslant 2$, i.e., iff the triangles of size 1 around them intersect. The following lemma is easy to verify.

Lemma 2.2. (i) If two triangles I_1, I_2 intersect, then there is a triangle I of size $\text{span}(I_1) + \text{span}(I_2)$ containing $I_1 \cup I_2$.

(ii) If two sets A_1, A_2 have neighboring points and A_j is contained in $D(I_j, 1/3)$ for triangles I_j, then there is a triangle I of size $\text{span}(I_1) + \text{span}(I_2)$ such that $A_1 \cup A_2$ is contained in $D(I, 1/3)$.

Proof of Lemma 1.1. Let S be a connected subset of $\mathbf{Z}^2$ with the property that $(R^+)^{2i}(S)$ is not empty. We have to give a lower bound on the set $(R^+)^i(S)$.

It is easy to verify the following commutation property of the rules R and Q:

$$QR(S) \subset RQ(S)$$

It follows that

$$(R^+)^i(S) \subset Q^i R^{2i}(S)$$

If $(R^+)^{2i}(S)$ is not empty, then $\text{span}(S, 1/3) > 2i$. It follows from Lemma 2.1 that $R^{2i}(S)$ is not empty. The set $Q^i(R^{2i}(S))$ then contains a full triangle of span i, which contains $(i+1)(i+2)/2$ elements. ∎

3. THE MAIN RESULT

3.1. The Effect of Toom's Rule on Components

Suppose that the set S consists of the connected components $S_1,..., S_n$. Connectedness is understood here in the graph $\mathbf{G}$. The next statement shows that Toom's rule does not break up or connect components. More precisely, it implies that the components of $R(S)$ are the nonempty ones among the sets $R(S_i)$. This statement will not be used directly, but is useful for getting some feeling for the way Toom's rule acts.

Fact 3.1. Let S be a subset of $\mathbf{W}$.

(a) If S is connected then $R(S)$ is connected or empty.

(b) If E is a connected subset of $R(S)$, then $S \cap Q(E)$ is connected.

Proof. Proof of (a). Let a and b be two points in $R(S)$. Let a_1 be a point of S in $Q(a)$, and b_1 a point of S in $Q(b)$. These points are connected in S by a path. Each edge of the path is contained in exactly one tile held by S. We have obtained a path of tiles connecting the tile with center a to the tile with center b. The centers of these tiles form a path connecting a and b in $R(S)$.

Proof of (b). Let a, b be two points in $S_0 = S \cap Q(E)$. We have to find a path in S_0 connecting them. Since the set E is connected, it is enough to find such a path when a, b are in two neighboring tiles, and then work step by step. If the intersection point of the two neighbor tiles is in S_0, then a, b are clearly connected through it. Otherwise, S_0 contains the edge in both tiles opposite the intersection. It is easy to see from Fig. 2 that these two edges have an edge of $\mathbf{G}$ connecting them. ▮

3.2. Cuts and Thickness

For a subset A of S, let $\mathbf{b}_S(A)$ be the set of all elements of $S \backslash A$ that are neighbors of an element of A.

The triple (C, A_1, A_2) of disjoint subsets of a connected set S in $\mathbf{W}$ is called a *cut of S with parameters* $|C|$, m if every path in S from A_1 to A_2 passes through an element of C, and

$$m = \min_{j=1,2} \operatorname{Span}(A_j \cup C)$$

If there is no path from A_1 to A_2, then $(\varnothing, A_1, A_2)$ is a cut. The cut is called *closed* if

$$(\mathbf{b}_S(A_1) \cup \mathbf{b}_S(A_2)) \subset C$$

Generally, our constructions will yield a cut (C, A_1, A_2) that is not necessarily closed. It can be made closed by adding to A_j all elements of S reachable from A_j on paths without passing through C. This operation does not increase the cutting set, but increases the sets A_j. A cut is *connected* if both sets $A_j \cup C$ for $j = 1, 2$ are connected.

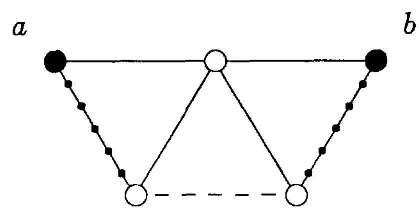

Fig. 2. For the proof of Fact 3.1(b).

Let $\Theta(S, \alpha)$ (the *α-thickness* of S) denote the smallest number k such that S has a (not necessarily connected) cut with parameters k, m with $m > \alpha k$. If no such k exists, then the α-thickness is ∞. If the α-thickness of a large set S is k, then a set of cardinality $< k$ cannot cut off from S a subset of span $> \alpha k$, i.e., the set S does not have large parts connected to the main body only on thin bridges. The main result is the following theorem, showing that the rule

$$R^+ = Q \circ R^2$$

increases the thickness (Fig. 3).

Theorem 3.1 (Main Theorem). We have

$$\Theta(R^+(S), 6) \geqslant \Theta(S, 6) + 1$$

As an example, let us look at the set on Fig. 3 before and after the application of the rule R^+. The narrow connection between the two parts becomes wider.

3.3. Auxiliary Notions of Thickness

The rule R itself does not increase the thickness of a set. It cannot even be said that the thickness is preserved. Though connections between large parts of the set do not seem to become narrower, some of these parts may become larger, as the example in Fig. 4 shows. In this example, the three thin connections holding the central reversed triangle do not become thicker, but this reversed triangle becomes bigger.

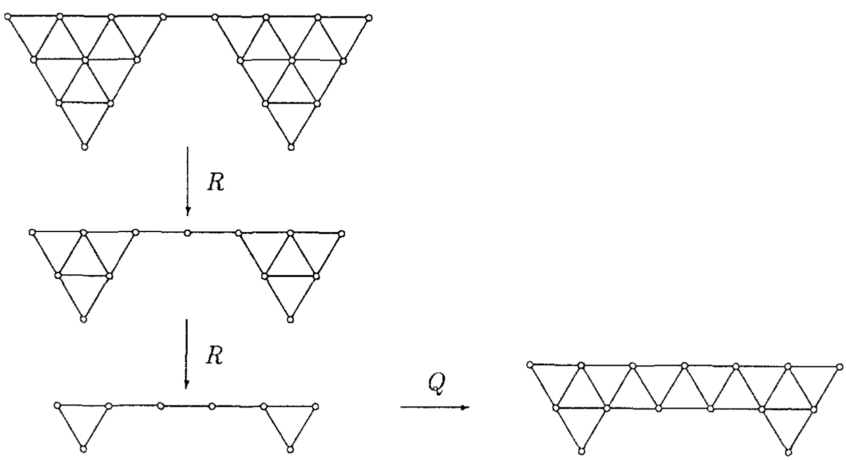

Fig. 3. The rule R^+ increases thickness.

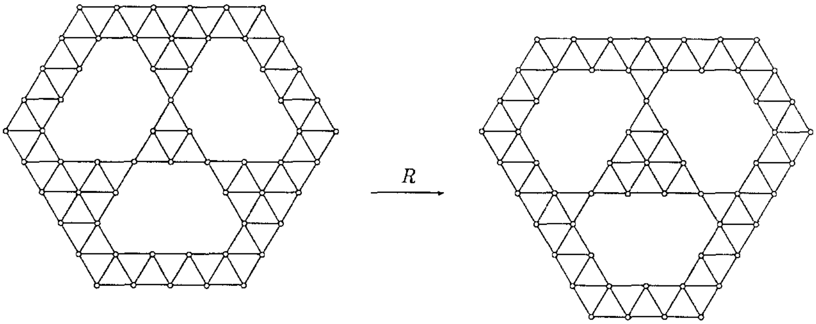

Fig. 4. The rule R may have an adverse effect on thickness.

To take these adverse effects into account, we need an auxiliary notion. Let $\vartheta(S, \alpha, \beta)$ [the (α, β)-*thickness* of S] denote the smallest number k such that S has a connected cut with parameters k, m with $m > \alpha k + \beta$. Notice that the difference is not only in the extra argument β, but in that it deals only with connected cuts. Its relation to $\Theta(S, \alpha)$ is shown by the following theorem.

Theorem 3.2.

$$\Theta(S, \alpha) = \vartheta(S, \alpha, 0)$$

Before proving this theorem, we need the following lemma.

Lemma 3.1. Let (C, A, B) be a closed cut of S with $|C| < \vartheta(S, \alpha, \beta)$. Let us break $A \cup C$ into components $U_1, U_2, \ldots$, and $B \cup C$ similarly into components $V_1, V_2, \ldots$. Then we have either

$$\mathrm{Span}(U_i) \leqslant \alpha |U_i \cap C| + \beta$$

for all i, or

$$\mathrm{Span}(V_j) \leqslant \alpha |V_j \cap C| + \beta$$

for all j.

Proof. Suppose that the first relation does not hold. Without loss of generality, let us assume that

$$\mathrm{Span}(U_1) > \alpha |U_1 \cap C| + \beta \tag{3.1}$$

Let j be arbitrary. Let $C' = U_1 \cap V_j$. Then $C' \subset C$. Let $A' = U_1 \backslash C'$, $B' = V_j \backslash C'$.

The triple (C', A', B') is a connected, closed cut of S. The connectedness follows immediately from the definition. To show that it is a closed cut, we have to show $\mathbf{b}_S(A') \subset C'$. The relation $A' \subset A$ implies $\mathbf{b}_S(A') \subset A \cup \mathbf{b}_S(A) \subset A \cup C$, and hence, since $A' \cup C'$ is a component of $A \cup C$, we have $\mathbf{b}_S(A') \subset C'$.

It follows from the fact that (C', A', B') is a connected cut and from $\vartheta(S, \alpha, \beta) > |C|$ that

$$\min(\mathrm{Span}(U_1), \mathrm{Span}(V_j)) \leqslant \alpha |C'| + \beta$$

This, together with (3.1), implies

$$\mathrm{Span}(V_j) \leqslant \alpha |V_j \cap C| + \beta \quad \blacksquare$$

Proof of Theorem 3.2. Let S be a set with $\vartheta(S, \alpha, 0) > k$. We will estimate $\Theta(S, \alpha)$. Let (C, A, B) be a closed cut of S with $|C| = k$. Let us break $A \cup C$ into components $U_1, U_2, \ldots$, and $B \cup C$ similarly into components $V_1, V_2, \ldots$. Then, Lemma 3.1 says that we have either

$$\mathrm{Span}(U_i) \leqslant \alpha |U_i \cap C| \tag{3.2}$$

for all i, or

$$\mathrm{Span}(V_j) \leqslant \alpha |V_j \cap C|$$

for all j. Without loss of generality, assume that (3.2) holds. Then we have

$$\mathrm{Span}(A \cup C) \leqslant \sum_i \mathrm{Span}(U_i) \leqslant \alpha |C| \quad \blacksquare$$

When $\beta > 0$ then the relation between our notion of thickness defined (for technical reasons to become clear later) with connected cuts and the notion defined with arbitrary cuts is not as simple as above. The reason can be seen from the last summation in the above proof. If we had $\alpha |U_i \cap C| + \beta$ instead of $\alpha |U_i \cap C|$, then the summation would bring in $n\beta$, where n is the number of terms.

3.4. Outline of the Proof of the Main Theorem

The following theorem, to be proved later, shows that the original Toom rule "almost" preserves thickness.

Theorem 3.3. If $\beta \leqslant 3 \cdot \mathrm{defl} - 3$, then

$$\vartheta(R(S), \alpha, \beta + 2) \geqslant \vartheta(S, \alpha, \beta)$$

The following theorem, to be proved later, says that the rule Q increases thickness.

Theorem 3.4. Suppose that $\beta \leqslant 3 \cdot \text{defl} - 2$. Then,

$$\vartheta(Q(S), \alpha, \beta + 2 - \alpha) \geqslant \vartheta(S, \alpha, \beta) + 1$$

Proof of Theorem 3.1. We apply the above theorems to R, R, and Q, consecutively, with $\alpha = 6$ throughout, but with $\beta = 0, 2, 4$ in the three stages. ▮

4. THE EFFECT OF TOOM'S RULE ON THICKNESS

Proof of Theorem 3.3. Let $U = R(S)$. Let (C, A_1, A_2) be a connected cut of U with $|C| < k$. Without loss of generality, we can assume that it is a closed cut. Our goal is to estimate $\min_{j=1,2} \text{Span}(A_j \cup C)$. We will find a certain cut (C', B_1, B_2) of S.

For each element a of C, we define an element a' in $S \cap Q(a)$, and set $C' = \{a' : a \in C\}$. To define a', remember the notation e_i from (2.1). Let us group the neighbors of a in three connected pairs $P_i (i = 1, 2, 3)$, where (Fig. 5)

$$P_i(a) = \{b \neq a : e_i(a) \in Q(b)\}$$

The pair $P_i(a)$ consists of the centers of those tiles containing the corner $e_i(a)$. For each i, the pair P_i may intersect one of the sets A_j. It cannot intersect both, since A_1 and A_2 are separated by C.

1. Suppose that only one pair, say P_i, is intersected by A_1, and $e_i(a) \in S$. Then let $a' = e_i(a)$.

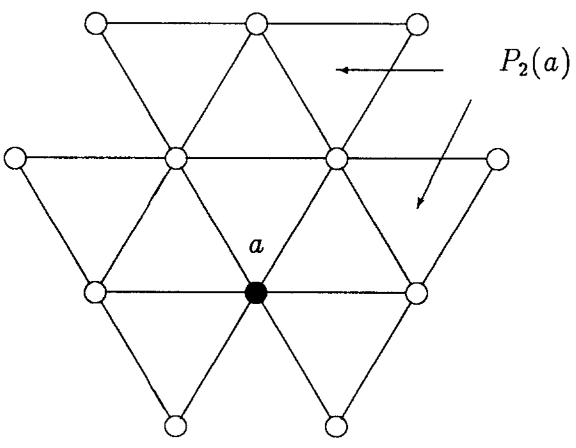

Fig. 5. The pairs of tiles with centers in $P_i(a)$.

2. Suppose that two pairs are intersected by A_1, and the third one, say P_i, is not, and $e_i(a) \in S$. Then let $a' = e_i(a)$.
3. In all other cases, we choose a' arbitrarily from the set $Q(a) \cap S$.

Now let

$$B_j = (S \cap Q(A_j)) \setminus C'$$

Lemma 4.1. The triple (C', B_1, B_2) is a cut.

Proof. It is enough to prove that if there is a path between some elements $b_j \in B_j$ for $j = 1, 2$, then this path passes through an element of C'. Let $b_1 = v_1, v_2, \ldots, v_n = b_2$ be such a path. For both $j = 1, 2$, the element b_j is contained in a tile $Q(a_j)$ for some $a_j \in A_j$. Let q be the last p such that $v_p \in Q(a)$ for some a in A_1. Let $Q(w)$ be the tile containing the pair $\{v_q, v_{q+1}\}$. Then $w \in C$, since (C, A_1, A_2) is a closed cut. It is easy to see from the definition above that w' is either v_q or v_{q+1}. ∎

Let us complete the proof of Theorem 3.3. We replace the cut (C', B_1, B_2) with the closed cut $(C', \bar{B}_1, \bar{B}_2)$, where $B_i \subset \bar{B}_i$. Let $U_1, \ldots, U_n$ be the components of $\bar{B}_1 \cup C'$, and $V_1, V_2, \ldots$ the components of $\bar{B}_2 \cup C'$. It follows from Lemma 3.1 that either

$$\operatorname{Span}(U_i) \leqslant \alpha |U_i \cap C'| + \beta \tag{4.1}$$

for all i, or

$$\operatorname{Span}(V_j) \leqslant \alpha |V_j \cap C'| + \beta$$

for all j. Let us suppose without loss of generality that (4.1) holds. It follows from the definition of C' and B_1 that the tile $Q(a)$ intersects $B_1 \cup C'$ for all $a \in A_1 \cup C$. Let

$$W_i = \{a \in A_1 \cup C : Q(a) \cap U_i \neq \varnothing\}$$

$$U_i' = Q(W_i)$$

Then $\bigcup_i W_i = A_1 \cup C$, $U_i \subset U_i'$. It follows from the connectedness of U_i that $\operatorname{span}(U_i, \operatorname{defl}) = \operatorname{span}(I_i)$ for a triangle I_i such that $U_i \subset D(I_i, \operatorname{defl})$. Then the triangle $J_i = D(I_i, \operatorname{defl} - 1)$ contains U_i', and the triangle $R(J_i)$ contains W_i. Let $K_i = D(R(J_i), -1/3)$, i.e., the blowup of $R(J_i)$ by 1/3.

Let us call the sets W_i, W_j *neighbors* if they either intersect or have neighboring elements. It follows from the connectedness of $\bigcup_i W_i$ that the set $\{W_1, W_1, \ldots\}$ is connected under this neighbor relation. We constructed K_i in such a way that $W_i \subset D(K_i, 1/3)$. Therefore, if W_i and W_j are neighbors, then K_i and K_j intersect. Let us call two triangles K_i, K_j

neighbors if they intersect. Then, from the fact that the set $\{W_1, W_2, \ldots\}$ is connected under the neighbor relation, it follows that the set $\{K_1, K_2, \ldots\}$ is also connected under its neighbor relation.

According to Lemma 2.2, if triangles I, J intersect, then there is a triangle containing their union whose span is $\leqslant \mathrm{span}(I) \cup \mathrm{span}(J)$. It follows that there is a triangle K containing $\bigcup_i K_i$ such that $\mathrm{span}(K) \leqslant \sum_i \mathrm{span}(K_i)$. As we know, $\mathrm{span}(K_i, d) = \mathrm{span}(J_i) + 3d - 1$ for any non-negative d. It follows from (4.1) that

$$\begin{aligned}\mathrm{span}(K_i, 1/3) &= \mathrm{span}(J_i) + 1 - 1 = \mathrm{span}(I_i) - 3(\mathrm{defl} - 1)\\ &\leqslant \alpha\,|U_i \cap C'| + \beta - 3 \cdot \mathrm{defl} + 3\end{aligned}$$

We have therefore

$$\begin{aligned}\mathrm{span}(K) &\leqslant \alpha \sum_i |U_i \cap C'| + n(\beta - 3 \cdot \mathrm{defl} + 3)\\ &\leqslant \alpha\,|C'| + n(\beta - 3 \cdot \mathrm{defl} + 3)\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{span}(A_1 \cup C, \mathrm{defl}) &\leqslant \mathrm{span}(K, \mathrm{defl} - 1/3)\\ &\leqslant a\,|C'| + n(\beta - 3 \cdot \mathrm{defl} + 3) + 3 \cdot \mathrm{defl} - 1\\ &\leqslant \alpha\,|C| + \beta + 2\end{aligned}$$

where we used the assumption $\beta \leqslant 3 \cdot \mathrm{defl} - 3$ to imply that the coefficient of n is not positive; therefore, we can replace n with 1. ▮

5. THE EFFECT OF INFLATION ON THICKNESS

Proof of Theorem 3.4. For a subset E of $Q(S)$, let

$$Q^{-1}(E, S) = \{a \in S \colon Q(a) \cap E \neq \varnothing\}$$

Suppose that (C, R_1, R_2) is a connected cut of $Q(S)$ with $|C| \leqslant \vartheta(S, \alpha, \beta)$. Without loss of generality, we can assume that it is a closed cut. Our goal is to estimate $\min_{j=1,2} \mathrm{Span}(R_j)$. From the fact that R_1, R_2 are separated by a cut, it follows that the sets $Q^{-1}(R_j, S)$ are disjoint. Let

$$S_j = Q^{-1}(R_j, S)$$

Lemma 5.1. We have

$$R_j \subset Q(S_j) \subset R_j \cup C$$

for $j = 1, 2$.

Proof. The first relation follows immediately from the definition. For the second relation, note that

$$Q(S_j) \subset R_j \cup \mathbf{b}_S(R_j)$$

which is contained in $R_j \cup C$ by the closedness of the cut (C, R_1, R_2). ∎

Now we proceed similarly to the proof of Theorem 3.3. However, we are trying to make the new cutting set C' smaller than the old one.

Lemma 5.2. Let us use the notation introduced above. There is an element x of C and a mapping $a \to a'$ defined on $C \backslash \{x\}$ such that we have $a \in Q(a')$, and with

$$C' = \{a' : a \neq x\}, \qquad S'_j = S_j \backslash C'$$

the triple (C', S'_1, S'_2) is a cut of S.

The proof of this lemma is left to the next section.

Now we conclude the proof of Theorem 3.4 analogously to the end of the proof of Theorem 3.3. Let $(C', \bar{S}'_1, \bar{S}'_2)$ be closed cut such that $S'_j \subset \bar{S}'_j$. Let $U_1, U_2,\ldots$ be the components of $\bar{S}'_1 \cup C'$, and $V_1, V_2,\ldots$ the components of $\bar{S}'_2 \cup C'$. It follows from Lemma 3.1 that either

$$\mathrm{Span}(U_i) \leqslant \alpha |U_i \cap C'| + \beta \tag{5.1}$$

for all i, or

$$\mathrm{Span}(V_j) \leqslant \alpha |V_j \cap C'| + \beta$$

for all j. Let us suppose without loss of generality that (5.1) holds. Let $W_i = Q(U_i)$. Let us remember the superfluous element x, and define $W_0 = \{x\}$. It follows from our construction that

$$R_1 \cup C \subset \bigcup_i W_i$$

It follows from the connectedness of U_i that $\mathrm{Span}(U_i) = \mathrm{Span}(U_i, \mathrm{defl}) = \mathrm{span}(I_i)$ for a triangle I_i such that $U_i \subset J_i = D(I_i, \mathrm{defl})$. Then $W_i \subset Q(J_i)$. Let $K_i = D(Q(J_i), -1/3)$ for $i > 0$, and $D(\{x\}, -1/3)$ for $i = 0$. Just as in the proof of Theorem 3.3, we can conclude that there is a triangle K containing $\bigcup_i K_i$ such that $\mathrm{span}(K) \leqslant \sum_i \mathrm{span}(K_i)$. It follows from (5.1) that, for $i > 0$,

$$\begin{aligned} \mathrm{span}(K_i) &= \mathrm{span}(J_i) + 1 + 1 = \mathrm{span}(I_i) - 3 \cdot \mathrm{defl} + 2 \\ &\leqslant \alpha |U_i \cap C'| + \beta - 3 \cdot \mathrm{defl} + 2 \end{aligned}$$

We have therefore

$$\operatorname{span}(K) \leqslant \alpha \sum_i |U_i \cap C'| + n(\beta - 3 \cdot \text{defl} + 2) + \operatorname{span}(K_0)$$

$$\leqslant \alpha |C'| + n(\beta - 3 \cdot \text{defl} + 2) + 1$$

Finally,

$$\begin{aligned} \operatorname{span}(R_1 \cup C, \text{defl}) &\leqslant \operatorname{span}(K, \text{defl} - 1/3) \\ &\leqslant \alpha |C'| + n(\beta - 3 \cdot \text{defl} + 2) + 1 + 3 \cdot \text{defl} - 1 \\ &\leqslant \alpha(|C| - 1) + \beta + 2 = \alpha |C| + \beta + 2 - \alpha \end{aligned}$$

where we used the assumption $\beta \leqslant 3 \cdot \text{defl} - 2$ to imply that the coefficient of n is not positive; therefore, we can replace n with 1. ▮

6. CUTTING THE PREIMAGE WITH FEWER POINTS

6.1. Conditions for a Cut in the Preimage

Proof of Lemma 5.2. In later parts of the proof, we will give an algorithm for the definition of the distinct elements $a_1, a_2, \ldots$, the number s with $a_s = x$, and the sets

$$C'_t = \{a'_i : i \leqslant t, i \neq s\}$$

Let $S'_i = S_i \backslash C'_t$. Let $C'_0 = \varnothing$. Assume that $a_1, \ldots, a_t$ and C'_{t-1} have already been defined. First we see that, given $a_1, a_2, \ldots, a_t$, what conditions must be satisfied by s and a'_t to make (C', S'_1, S'_2) a cut of S.

The element a_t is contained in three tiles $Q(b_i(t))$ for $i = 1, 2, 3$ (Fig. 6). They are numbered in such a way that

$$a_t = e_i(b_i(t))$$

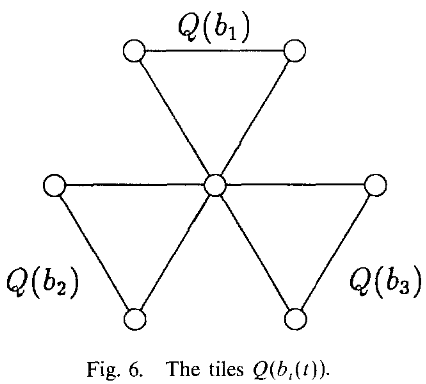

Fig. 6. The tiles $Q(b_i(t))$.

Let us write $B(t) = \{b_1(t), b_2(t), b_3(t)\}$. We say that a_t is *superfluous* if one of the S_j^{t-1} does not intersect the set $B(t)$. We will choose $a_1, a_2,...$ later in such a way that there is a t such that a_t is superfluous.

Condition 6.1. The point a_s is the first superfluous element of the sequence $a_1, a_2,...$.

If a_t is not superfluous, then there are a b and a j such that

$$\{b\} = B(t) \cap S_j^{t-1}$$

Such a b is called *eligible* for t. Let $E(t)$ be the set of those (one or two) elements of $B(t)$ that are eligible for t.

Condition 6.2. If a_t is not superfluous, then $a'_t \in E(t)$.

Lemma 6.1. If Conditions 6.1 and 6.2 are satisfied, then (C', S'_1, S'_2) is a cut of S.

Proof. Suppose that there is a path $u_1,...,u_n$ going from S'_1 to S'_2 in S. Let u_p be the first element of the path that is not in S'_1. We will prove that it is in C'. The point a in the intersection of $Q(u_{p-1})$ and $Q(u_p)$ is the neighbor of an element of R_1, since it is in $Q(u_{p-1})$. If it is an element of R_1 itself, then $u_p \in S_1$. Since $u_p \notin S'_1$, it follows that $u_p \in C'$ and we are done.

Suppose therefore that $a \notin R_1$. Then $a \in C$, since (C, R_1, R_2) is a closed cut. Let t be such that $a = a_t$. Then $u_{p-1} \in S_1^{t-1}$. If $u_p \notin C'_{t-1}$, then $u_p \in S_2^{t-1}$, by the definition of S_2. Then a_t is not superfluous, and by Condition 6.2, a'_t is either u_{p-1} or u_p. ∎

6.2. The Choice of a'_t and a_{t+1}

After Lemma 6.1, what is left from Lemma 5.2 to prove is that the sequences a_t, a'_t can be chosen satisfying Condition 6.2 in such a way that one of the a_t is superfluous.

The construction will contain an appropriately chosen constant $r = 1, 2,$ or 3. If

$$a_{t-1} \in Q(b_r(t)) \tag{6.1}$$

then we say that a *forward choice* is made at time t. In this case, a_t is in corner r of the tile containing both a_t and a_{t-1}. We call this tile the *backward* tile (Fig. 7). The value of the linear function L_r is greater on a_{t-1} than on a_t. Let us call the two other tiles containing a_t the *forward* tiles.

The set

$$F(t) = B(t) \cap (S_1^{t-1} \cup S_2^{t-1}) \setminus \{b_r\}$$

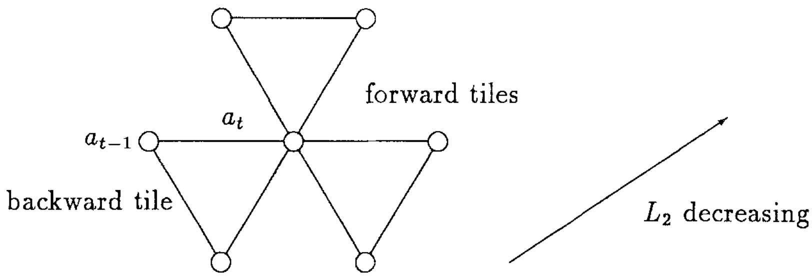

Fig. 7. Backward and forward tiles, with $r = 2$.

is the set of the centers of one or two forward tiles for t. In the case of a forward choice, the corner r of one of the forward tiles is chosen for a_{t+1}. Suppose that there is a b in $F(t)$ satisfying

$$e_r(b) \in C \setminus \{a_1, ..., a_t\} \tag{6.2}$$

Then, choosing a_{t+1} as such a b would make a strong forward choice.

If, in addition to (6.1), we also have $a_{t+1} = e_r(a'_t)$, then we say that a *strong forward choice* is made.

Condition 6.3. Suppose that there is a b in $E(t) \cap F(t)$ satisfying (6.2). Then a_{t+1} is such a b, and with $a'_t = b$ a strong forward choice is made.

Conditions 6.2 and 6.3 are the only ones restricting the choice of a'_t and a_{t+1} for $t > 1$. Otherwise, the choice is arbitrary.

Lemma 6.2. Suppose that no superfluous a_i was found for $i = 1, ..., t$, all earlier choices (if any) were forward, and

$$F(t) \cap S_j^{t-1} \neq \varnothing \qquad \text{for} \quad j = 1, 2 \tag{6.3}$$

Then there is a b in $F(t)$ satisfying (6.2) and therefore a forward choice can be made. If there is a b in $E(t) \cap F(t)$ satisfying (6.2), then all choices beginning with t are strongly forward, until a superfluous node is found.

Proof. By the assumption (6.3), the elements of $F(t)$ are contained in two different sets S_j. It follows from Lemma 5.1 that the two forward tiles are contained in different sets $R_j \cup C$. There is an edge between the corners r of the two forward tiles. Since C separates R_j, it must contain one of these points $e_r(b)$. Since all our earlier choices were forward, the function L_r is strictly decreasing on the sequence $a_1, a_2, ..., a_t, e_r(b)$. Therefore, it is not possible that $e_r(b)$ is equal to one of the earlier elements of the sequence, and hence (6.2) is satisfied.

If a b in $F(t) \cap E(t)$ can be found satisfying (6.2), then according to Condition 6.3, the strong forward choice $a'_t = b$, $a_{t+1} = e_r(b)$ is made. From $a'_t \notin S_1^t \cup S_2^t$, it follows that either a_{t+1} is superfluous or $E(t+1) = F(t+1) = B(t+1) \setminus \{a'_t\}$. In the latter case, the conditions of the present lemma are satisfied for $t+1$, implying that the next choice is also forward, etc. ∎

6.3. The Choice of r, a_1, a'_1, and a_2

Condition 6.4. (1) If a_1 can be chosen superfluous, then it is chosen so.

(2) If a_1 cannot be chosen superfluous, but it can be chosen to make $|E(1)| > 1$, then it is chosen so. In this case, r is chosen to make $E(1) = F(1)$.

If the second case of the above condition occurs, then all conditions of Lemma 6.2 are satisfied with $t = 1$.

Condition 6.5. Suppose that none of the choices of Condition 6.4 is possible, and r, a_1, a'_1, and a_2 can be chosen to either make a_2 superfluous or to satisfy the conditions of Lemma 6.2 with $t = 2$. Then they are chosen so.

Lemma 6.3. The elements r, a_1, a'_1, and a_2 can always be chosen in such a way that either Condition 6.4 or Condition 6.5 applies.

Before giving the proof of this lemma, let us finish, with its help, the proof of Lemma 5.2. The complete algorithm of choosing a_t, a'_t, and r is as follows. Choose a_1 to satisfy Condition 6.4. If the second part applies, then choose r accordingly. If Condition 6.5 applies, then choose r, a'_1, and a_2 to satisfy Conditions 6.2 and 6.5. From now on, choose a'_t, a_{t+1} to satisfy Conditions 6.2 and 6.3.

A superfluous a_t will always be found. Indeed, if the first part of Condition 6.4 applies, then a_1 is superfluous. If the second part applies, then the conditions of Lemma 6.2 are satisfied with $t = 1$. If Condition 6.5 applies, then they are satisfied with $t = 2$. From this time on, strong forward choices can be made until a superfluous a_t is found. This is unavoidable, since C is finite and hence we cannot go on making strong forward choices forever. ∎

Proof of Lemma 6.3. Suppose that the statement of the lemma does not hold. We will arrive at a contradiction. Choose a_1 arbitrarily. We have $|E(1)| = 1$. We can choose r to get $|F(1)| = 2$, $E(1) \subset F(1)$. We will show that we can then make a forward choice (not strong) for each t and

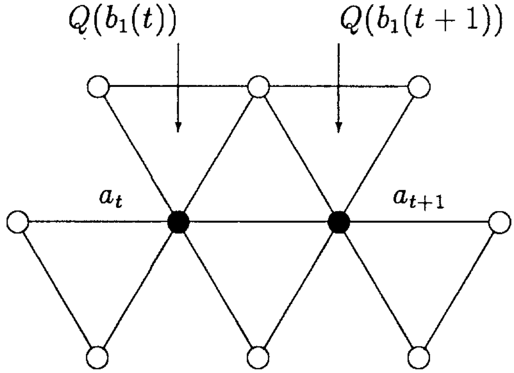

Fig. 8. For the proof of Lemma 6.3. The shaded tiles $Q(b_1(t))$ and $Q(b_1(t+1))$ belong to S_1.

recreate the conditions (6.3) indefinitely. This is the desired contradiction, since our set is finite.

Assume that we succeeded until t. By Lemma 6.2, there is a b in $F(t)$ such that (6.2) holds. If $b \in E(t)$, then with the choice $\bar{a}_1 = a_t$, $\bar{a}'_1 = b$, and $\bar{a}_2 = e_r(b)$, Condition6.5 would apply, and we assumed this is impossible. Therefore $b \notin E(t)$.

Without loss of generality, let us assume

$$E(t) = \{b_1(t)\} \subset S_1^{t-1}, \qquad r = 2$$

Then $b \neq b_1(t)$. From $b \in F(t)$, it follows that $b \neq b_2(t)$, hence $b = b_3(t)$. Since a_t is not superfluous, the assumption $E(t) = \{b_1(t)\}$ implies

$$B(t) \cap S_1^{t-1} = \{b_1(t)\}, \; B(t) \cap S_2^{t-1} = \{b_2(t), b_3(t)\}$$

Let us show $b_1(t+1) \in S_1^{t-1}$. It is easy to check that the two tiles $Q(b_1(t))$ and $Q(b_1(t+1))$ intersect in $a = e_2(b_1(t)) = e_3(b_1(t+1))$ (Fig. 8). If $b_1(t+1)$ belonged to S_2^{t-1}, then, by Lemma 5.1, the tile $Q(b_1(t+1))$ would be contained in $R_2 \cup C$, while, for a similar reason, the tile $Q(b_1(t))$ is contained in $R_1 \cup C$. Then the intersection point a would have to belong to C. But then we could satisfy (6.2) with $b = b_1(t) \in E(t)$.

We have $b_3(t+1) \in S_2^{t-1}$. Indeed, if it belonged to S_1^{t-1}, then the choice $\bar{a}_1 = a_{t+1}$, $\bar{a}_2 = a_t$ would again satisfy all conditions of Lemma 6.2, which we supposed is impossible. We found that the neighborhood of a_{t+1} is just a shift of the neighborhood of a_t. The could continue indefinitely. ∎

7. CONCLUSION

Let us make a remark on the possible extension of the present work. The presence of failures seems to necessitate a more complicated notion of thickness, and it is not clear what the appropriate generalization of the main theorem should be in that case.

A variant of the main theorem can probably be proven where the size of the cutting set is measured in terms of its span instead of number of elements. If the proof of that variant is significantly simpler, then it should replace the present theorem.

The stability property of the rules analogous to Toom's rules can also be proved for continuous-time systems. In such systems, the transition rule is not applied simultaneously at all sites, but each site applies it at random times. It seems that the consensus property of slightly biased Toom rules holds also for this situation. Though the methods used in the present paper seem to depend on synchrony, especially the fact that the inflation operation is carried out all at once, it is hoped that the concepts will be useful in extensions to these related problems.

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